Exponential Period

of Neuronal Recurrence Automata with Excitatory Memory

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RÉSUMÉ. Nous infirmons la conjecture sur les équations neuronales récurrentes à mémoire excitatrice en montrant que la longueur du cycle peut être exponentielle par rapport à la taille de la mémoire.

ABSTRACT. We disprove the conjecture on recurrence automata with excitatory memory by showing that the cycle length can grow exponentially respectively to the size of the memory.

MOTS-CLÉS : Coefficients d'interaction , période.

KEYWORDS : Weighting coefficients, period.

1. Introduction

In [1] it is suggested that the dynamic behavior of a single neuron with memory length k that does not interact with other neurons can be modeled by the following neuronal recurrence equation :

$$x(n) = \mathbf{1}\Big(\sum_{i=1}^{k} a_i x(n-i) - \theta\Big) \qquad (1)$$

where we have the following.

- $x(n)$ is a variable representing the state of the neuron at time $t = n$, $x(n) \in \{0, 1\}$.
- k is the memory length, that is to say, the state of the neuron at time $t = n$ depends on the state assumed by the neuron at the k previous steps $t = n-1, \ldots, n-k$.
- the values a_i $(i = 1, \ldots, k)$ are real numbers called the *weighting coefficients* ; a_i represents the influence of the state of the neuron at time $n - i$ on the state of the neuron at time n. That influence is said to be *excitatory* if $a_i > 0$, *inhibitory* if $a_i < 0$ and *null* if a_i is equal to zero.
- θ is a real number called the *threshold*.
- $\mathbf{1}[u] = 0$ if $u < 0$, and $\mathbf{1}[u] = 1$ if $u \geq 0$.

The system obtained by interconnecting several neurons is called a Neural Network (NN). Such networks were introduced in [4], and are quite powerful. Indeed, it can be shown that they can be used to simulate any Turing machines. More recently, NN have been studied extensively as tools for solving various problems such as classification, speech recognition, and image processing. The application field of the threshold functions is large. The spin moment of the spin glass is one of the widest example in solid state physics that have been simulated by NN. In electronics, for instance, a threshold function represents a transistor ; in social science a threshold function is often used to represent vote laws.

Let p and T be two positive integers such that $p > 0$ and $T \geq 0$. Equation (1) is said to be of period p and transient T if and only if :

- $Y(p+T) = Y(T)$
- $\nexists T'$ and p' $(T', p') \neq (T, p)$ $T \geq T'$ and $p \geq p'$ such that $Y(p'+T') = Y(T')$

where $Y(t) = (x(t), x(t-1), \ldots, x(t-k+2), x(t-k+1))$. The period and transient of sequences generated by a neuron are good measures of the complexity of the behavior of the neuron.

We are interested in the longest period $LP(k)$ that can be generated by a neuronal recurrence equation with memory k. In [3], it was conjectured that if $(a_i)_{1\leq i\leq k} \in \mathbb{R}$, then $LP(k) \leq 2k$. This conjecture has been disproved. The best known lower bound in $LP(k)$

is $O(e^{\sqrt{k\ln(k)}})$ and it was proved in [6].
When all the weighting coefficients are positive, the influence of the previous states of a neuron (at time $n-k, n-k+1, \ldots, n-2, n-1$) on its state at time n is excitatory, and from a physiological point of view, it is important to know the behavior of that class of neuron. In [3], it was also conjectured that if $\forall\, i, i = 1, \ldots, k,\ a_i \in \mathbb{R}^+$ (i.e $a_i \geq 0$), then $LP(k) \leq k$. This conjecture has been disproved in [5] where a neuronal recurrence equation of memory length k and of period $O(k^3)$ has been exhibited.
In this paper, we exhibit a neuronal recurrence equation of memory length h where all the weighting coefficients are strictly positive that generates a sequence of period $\Omega(e^{\sqrt[3]{h(\ln(h))^2}})$, this results more strongly contradicts the conjecture than the aforementioned counter example [5].

2. Neuronal Recurrence Equation with Positive Weighting Coefficients

Let k be a positive integer. For a vector $a \in \mathbb{R}^k$, a real number $\theta \in \mathbb{R}$ and a vector $z \in \{0,1\}^k$, we define the sequence $\{x(n) \ :\ n \in \mathbb{N}\}$ by the following recurrence :

$$x(t) = \begin{cases} z(t) & t \in \{0, \ldots, k-1\} \\ \mathbf{1}\left(\sum_{i=1}^{k} a_i x(t-i) - \theta\right) & t \geq k \end{cases} \qquad (2)$$

We denote by $S(a, \theta, z)$ the sequence generated by equation (2) and $T(a, \theta, z)$ its period.
Let m be a positive integer greater than 1, we denote the cardinality of the set $\mathcal{P} = \{p \ :\ p\ prime \text{ and } 2m < p < 3m\}$ by $\rho(m)$. Let us denote $p_1, \ldots, p_{\rho(m)}$ the prime numbers laying in $\{2m+1,\ 2m+2, \ldots, 3m-2, 3m-1\}$ and the sequence $\{\alpha_i \ :\ 1 \leq i \leq \rho(m)\}$ is defined as $\alpha_i \ =\ 3m - p_i,\ 1 \leq i \leq \rho(m)$.
It is easy to check that $\{2m+1,\ 2m+2, \ldots, 3m-2, 3m-1\}$ contains at most $\lceil \frac{m-1}{2} \rceil$ odd integers. It follows that

$$\rho(m) \ \leq\ \left\lceil \frac{m-1}{2} \right\rceil \qquad (3)$$

We set $k = (6m-1)\rho(m)$ and $\forall\, i \in \mathbb{N},\ 1 \leq i \leq \rho(m)$, we define :

$$\mu(m, \alpha_i) = \left\lfloor \frac{k}{3m - \alpha_i} \right\rfloor$$

$$\beta(m, \alpha_i) = k - ((3m - \alpha_i)\mu(m, \alpha_i))$$

From the previous definitions, we have $k = ((3m - \alpha_i)\mu(m, \alpha_i)) + \beta(m, \alpha_i)$.
It is clear that $\forall\, i \ \in\ \mathbb{N},\ 1 \leq i \leq \rho(m)$

$$2m + 1 \ \leq\ 3m - \alpha_i \ \leq\ 3m - 1$$

This implies
$$\frac{(6m-1)\rho(m)}{3m-1} \leq \frac{k}{3m-\alpha_i} \leq \frac{(6m-1)\rho(m)}{2m+1}$$

Therefore
$$2\rho(m) \; < \; \mu(m,\alpha_i) \; < \; 3\rho(m) \tag{4}$$

$\forall\, i \in \mathbb{N},\ 1 \leq i \leq \rho(m)$, we want to construct a neuronal recurrence equation $\{x^{\alpha_i}(n)\ :\ n \geq 0\}$ with excitatory memory of length k which evolves as follows :

$$\underbrace{00\ldots0}_{\beta(m,\alpha_i)}\underbrace{100\ldots0}_{3m-\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}\cdots\underbrace{100\ldots0}_{3m-\alpha_i}\cdots\underbrace{100\ldots0}_{3m-\alpha_i}\cdots \tag{5}$$

and which describes a cycle of length $3m-\alpha_i = p_i$.
$\forall\, i \in \mathbb{N},\ 1 \leq i \leq \rho(m)$, let $w^{\alpha_i} \in \{0,1\}^k$ be the vector defined by

$$w^{\alpha_i}(0)\ldots w^{\alpha_i}(k-1) = \underbrace{0\ldots0}_{\beta(m,\alpha_i)}\underbrace{\underbrace{10\ldots0}_{p_i}\cdots\underbrace{10\ldots0}_{p_i}}_{\mu(m,\alpha_i)p_i} \tag{6}$$

In other words, w^{α_i} is defined by :

$$w^{\alpha_i}(j) = \begin{cases} 1 & \text{if } \exists\, \ell,\ 0 \leq \ell \leq \mu(m,\alpha_i)-1 \text{ such that } j = \beta(m,\alpha_i)+\ell p_i \\ 0 & \text{otherwise} \end{cases}$$

Let γ be a real number satisfying $\gamma > 0$. We define the neuronal recurrence equation $\{x^{\alpha_i}(n)\ :\ n \geq 0\}$ by the following recurrence :

$$x^{\alpha_i}(t) = \begin{cases} w^{\alpha_i}(t) & t \in \{0,\ldots,k-1\} \\ \mathbf{1}\left(\sum_{j=1}^{k} \bar{a}_j x^{\alpha_i}(t-j) - \bar{\theta}\right) & t \geq k \end{cases} \tag{7}$$

where
$$\bar{a}_j = \begin{cases} \gamma & \text{if } j \in F \\ 0 & \text{if } j \in G \end{cases}$$

$$\begin{aligned} Pos(\alpha_i) &= \{jp_i : j = 1,\ldots,2\rho(m)\} \\ &= \{p_i, 2p_i, \ldots, (-1+2\rho(m))p_i, 2\rho(m)p_i\},\ \ 1 \leq i \leq \rho(m) \\ D &= \{i : i = 1,\ldots,k\} = \{1,2,\ldots,k-1,k\} \\ F &= \bigcup_{i=1}^{\rho(m)} Pos(\alpha_i) \\ G &= D \setminus F \\ \bar{\theta} &= \rho(m)\gamma \end{aligned}$$

By definition $Pos(\alpha_i)$ represents the set of indices j, $1 \leq j \leq k$ such that $x^{\alpha_i}(k-j) = 1$.
From the definition of $Pos(\alpha_i)$ and from equation (6), one can easily verify that

$$j \in Pos(\alpha_i) \implies x^{\alpha_i}(k-j) = 1 \tag{8}$$

$$j \in D \setminus Pos(\alpha_i) \implies x^{\alpha_i}(k-j) = 0 \tag{9}$$

$\forall\, d \in \mathbb{N},\ 0 < d < p_i$, we also denote $PPos(\alpha_i, d)$ the set of indices j such that $x^{\alpha_i}(k+d-j) = 1$, in other words :

$$PPos(\alpha_i, d) = \{\, j : x^{\alpha_i}(k+d-j) = 1 \text{ and } 1 \leq j \leq k \,\}$$

$\forall\, i, d \in \mathbb{N},\ 1 \leq i \leq \rho(m)$ and $0 < d < p_i$, we denote :

$$Q(\alpha_i, d) = \{d + jp_i : j = 0, 1, \ldots, \mu(m, \alpha_i)\},\quad 0 < d \leq \beta(m, \alpha_i)$$

$$Q(\alpha_i, d) = \{d + jp_i : j = 0, 1, \ldots, -1 + \mu(m, \alpha_i)\},\quad \beta(m, \alpha_i) < d < p_i$$

$$E(\alpha_i, d) = Q(\alpha_i, d) \cap F$$

$$z(\alpha_i, d) = card\ E(\alpha_i, d)$$

The neuronal recurrence equation $\{x^{\alpha_i}(n) \ :\ n \geq 0\}$ with excitatory memory of length k is defined by equation (6) and equation (7).
We will show that the neuronal recurrence equation $\{x^{\alpha_i}(n) \ :\ n \geq 0\}$ evolves as specified in equation (5).
In the following proposition, we present an important property.

Proposition 1 *$\forall\, i \in \mathbb{N}, 1 \leq i \leq \rho(m)$ and $\forall\, d \in \mathbb{N}, 1 \leq d < p_i$*

$$z(\alpha_i, d) \leq \rho(m) - 1$$

The following lemma characterizes the evolution of the sequence $\{x^{\alpha_i}(n) \ :\ n \geq 0\}$ at time $t = k$.

Lemma 1

$$x^{\alpha_i}(k) = 1$$

From Lemma 1 and equation (6), it is easy to verify that

$$PPos(\alpha_i, 1) = Q(\alpha_i, 1) \tag{10}$$

From the definition of $E(\alpha_i, 1)$, from equation (6), from equation (10) and from the Lemma 1, we check easily that :

$$\ell \in E(\alpha_i, 1) \implies x^{\alpha_i}(k+1-\ell) = 1 \text{ and } \tilde{a}_\ell = \gamma \tag{11}$$

$$\ell \in D \setminus E(\alpha_i, 1) \implies x^{\alpha_i}(k+1-\ell) = 0 \text{ or } \tilde{a}_\ell = 0 \tag{12}$$

The values of the sequence $\{x^{\alpha_i}(n) : n \geq 0\}$ at time $t = k+1, \ldots, k-1+p_i$ are given by the following lemma.

Lemma 2

$$\forall t \in \mathbb{N} \textit{ such that } 1 \leq t \leq 3m-1-\alpha_i, \textit{ we have } x^{\alpha_i}(k+t) = 0$$

It is easy to verify that $\forall i \in \mathbb{N},\ 1 \leq i \leq \rho(m)$, we have :

$$PPos(\alpha_i, j) = Q(\alpha_i, j)\ \forall j,\ 1 \leq j \leq 3m-1-\alpha_i$$

The following lemma characterizes the period of the sequence $\{x^{\alpha_i}(n) : n \geq 0\}$:

Lemma 3

There exists $\bar{a}, w^{\alpha_i} \in \mathbb{Z}^k$ *with* $a_j \geq 0$ *for every* $j = 1, \ldots, k$, *and* $\bar{\theta} \in \mathbb{R}$ *such that*

$$T(\bar{a}, \bar{\theta}, w^{\alpha_i}) = p_i$$

Proof
By application of Lemma 1 and Lemma 2 we deduce the result.
■
We showed that the recurrence neuronal equation $\{x^{\alpha_i}(n) : n \geq 0\}$ with excitatory memory of length k describes a cycle of length p_i and evolves as described in equation (5).
It is also shown that :

Lemma 4 *[2]*
If there is a neuronal recurrence equation with memory length k that generates sequences of periods $p_1, p_2, \ldots, p_r$, *then there is a neuronal recurrence equation with memory length* kr *that generates a sequence of period* $r \cdot \mathrm{lcm}(p_1, \cdots, p_r)$.

The previous lemma was amend so :

Lemma 5 *[7]*
If there is neuronal recurrence equation with memory length k *that generates a sequence* $\{x^j(n) : n \geq 0\}$, $1 \leq j \leq g$ *of transient length* T_j *and of period* p_j, *then there is a neuronal recurrence equation with memory length* kg *that generates a sequence of transient length* $g \cdot max(T_1, T_2, \ldots, T_g)$ *and period of length* $g \cdot \mathrm{lcm}(p_1, p_2, \ldots, p_g)$.

Now, we want to build a neuronal recurrence equation with excitatory memory of length $k\rho(m)$ which describes a cycle of length $\rho(m) \cdot lcm(p_1, p_2, \ldots, p_{\rho(m)})$
Let us denote $\lambda(m) = \prod_{i=1}^{\rho(m)} p_i$ and $h(m) = k\rho(m) = (6m-1)\cdot(\rho(m))^2$

Corollary 1 *For every positive integer m, $m \geq 2$, there exists $c \in \mathbb{R}^{h(m)}$, $\bar{\theta} \in \mathbb{R}$ and $v \in \{0,1\}^{h(m)}$ such that $c_i \geq 0$ for every $i = 1, \dots, h(m)$ and with $T(c, \bar{\theta}, v) = \rho(m)\lambda(m)$.*

Proof. From Lemma 3, we know that for $3m - \alpha_i \in \mathcal{P}$ we have that $T(\bar{a}, \bar{\theta}, w^{\alpha_i}) = 3m - \alpha_i$ with $\bar{a}_j \geq 0$ for every $j = 1, \dots, k$. We construct the vector c as in the fundamental lemma of composition of automata [2]. By construction, the vector c satisfies $c_i \geq 0$, for every $i = 1, \dots, h(m)$. From w^{α_i} with $3m - \alpha_i \in \mathcal{P}$, we construct v as in the fundamental lemma of composition of automata [2]. By application of Lemma 4 or by application by Lemma 5, we deduce that $T(c, \bar{\theta}, v) = \rho(m)\lambda(m)$.
■

The technique used in previous corollary defines several coefficients c_i as zero. We will

show that it is possible to modify the coefficients (c_i) so as to obtain all the coefficients being strictly positive.

Corollary 2 *For every positive integer m, $m \geq 2$ there exists $d \in \mathbb{R}^{h(m)}$, $\theta' \in \mathbb{R}$ and $v' \in \{0,1\}^{h(m)}$ such that $d_i > 0$ for every $i = 1, \dots, h(m)$ and with $T(d, \theta', v') = \rho(m)\lambda(m)$.*

Proof.
It suffices to apply Proposition 1 of [5] to Corollary 1.
■

From approximations of Rosser and Schoenfeld [8], we deduce :

Corollary 3

$$\prod_{2m < p < 3m,\ p \text{ prime}} p = \Omega(e^{\sqrt[3]{h(m)\,(\ln(h(m)))^2}})$$

Our main result is :

Theorem 1 *For every positive integer m there exists $d \in \mathbb{R}^{h(m)}$, $\theta' \in \mathbb{R}$ and $v' \in \{0,1\}^{h(m)}$ such that $d_i > 0$ for every $i = 1, \dots, h(m)$ and with $T(d, \theta', v') = \Omega(e^{\sqrt[3]{h(m)\,(\ln(h(m)))^2}})$.*

Proof.
From Corollary 3 and Corollary 2, we deduce the result.
■

Conclusion. The existence of a neuronal recurrence equation of memory length $h(m)$ which describes a cycle of length $\Omega(e^{\sqrt[3]{h(m)\,(\ln(h(m)))^2}})$ shows that the behavior of neuronal recurrence equations is complex when all the weighting coefficients are positive.

The technique used is inscribed in the framework of structural construction. Structural construction methods are the general and more powerful tools used in the study of sequences generated by neuronal recurrence equation [7, 9].

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3. Bibliographie

[1] E.R. Caianiello and A. De Luca. – *Decision Equation for Binary Systems : Applications to Neuronal Behavior*. – *Kybernetic* – **3**, (1966) 33-40.

[2] M. Cosnard, M.Tchuente and G. Tindo, *Sequences Generated by Neuronal Automata with Memory*, *Complex Systems*, **6**, (1992), p 13-20.

[3] M. Cosnard, D. Moumida, E. Goles and T.de.St. Pierre, *Dynamical Behavior of a Neural Automaton with Memory*, *Complex Systems*, **2**, (1988), 161-176.

[4] W. S. Mc Culloch and W. Pitts, *A Logical Calculus of the Ideas Immanent in Nervous Activity*, *Bulletin of Mathematical Biophysics*, **5**,(1943) 115-133.

[5] R. Ndoundam and M. Matamala, *Cyclic Evolution of Neuronal Automata with Memory when all the weighting coefficients are strictly positive*, *Complex Systems*, **12**,(2000), 379-390.

[6] R. Ndoundam and M.Tchuente, *Cycles exponentiels des réseaux de Caianiello et compteurs en arithmétique redondante*, *Technique et Science Informatiques*, **Volume 19** N° **7/2000**, 985-1008.

[7] R. Ndoundam and M.Tchuente, *Exponential transient length generated by a neuronal recurrence equation*, *Theoretical Computer Science* **306**(2003) 513-533.

[8] J.B. Rosser and L. Schoenfeld, *Approximate Formulas for some Functions of Prime Numbers*, *Illinois Journal of Mathematics*, **6** (1962), p. 64-94.

[9] M.Tchuente and G. Tindo *Suites générées par une équation neuronale à mémoire*, *Comptes Rendus de l'Académie des Sciences*, **tome 317, Série I**, 625-630, 1993.