A generalized finite difference method for the 2-D nonlinear shallow water equations

Ben Mansour Dia¹, ² – Ababacar Diagne¹, ³ – Leon M. S. Tine¹, ⁴

¹ Laboratoire d’Analyse Numérique et Informatique, Université Gaston Berger, Saint-Louis, BP 234, Sénégal.

² SRI - Center for Uncertainty Quantification in Computational Science & Engineering CEMSE, King Abdullah University of Sciences and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia
E-mail: benmansour.dia@kaust.edu.sa.

³ Division of Scientific Computing, Department of Information Technology, Uppsala University, Box 509 55 Uppsala, Sweden,
Email: ababacar.dian@it.uu.se.

⁴ Institut Camille Jordan, Université Claude Bernard Lyon 1,
Avenue Claude Bernard 69622 VILLEURBANNE cedex, France
E-mail: leon-matar.tine@univ-lyon1.fr.

RÉSUMÉ. Dans ce travail, nous présentons un schéma différences finies généralisé pour les équations de Saint-Venant en dimension deux. La discrétisation spatiale utilise la grille décalée C d’Arakawa. En plus du facteur d’implicitation θ, la discrétisation temporelle introduit un facteur de rapport de poids des noeuds spatiaux. L’analyse de stabilité tient compte de l’ordre de grandeur des paramètres. Nous discutons les propriétés du schéma et présentons quelques simulations.

ABSTRACT. In this paper, we propose a generalized finite difference method for the two-dimensional nonlinear shallow water equations. The space discretization uses the staggered grid C of Arakawa. Beside the implicit-explicit factor θ, the time discretization involves a balance ratio α of the spatial nodes. The stability analysis takes account the size of the parameters. We discuss the stabilizing properties of the scheme and present some numerical experiments.

MOTS-CLÉS : Différences finies, Equations de Saint-Venant, Analyse de stabilité
KEYWORDS : Finite difference, Shallow water equations, Stability analysis
1. Problem setting

Let $T$ be a real positive number standing for the study time, $\Omega$ represents a rectangular two-dimensional domain $\Omega = [0, L_1] \times [0, L_2]$. We denote $Q = [0, T] \times \Omega$ and consider the two-dimensional shallow water equations in the following form

\[
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} &= 0 & \text{in } Q, \\
\frac{\partial q_1}{\partial t} + gh \frac{\partial h}{\partial x} + q_1 \frac{\partial q_1}{\partial x} + q_2 \frac{\partial q_1}{\partial y} &= 0 & \text{in } Q, \\
\frac{\partial q_2}{\partial t} + gh \frac{\partial h}{\partial y} + q_1 \frac{\partial q_2}{\partial x} + q_2 \frac{\partial q_2}{\partial y} &= 0 & \text{in } Q, \\
h(t = 0, x, y) = h^0(x, y), (q_1, q_2) (t = 0, x, y) = (q_1^0, q_2^0) (x, y) & \text{in } \Omega, \\
\text{boundary conditions},
\end{aligned}
\]

where the variable $h$ designates the height of the water column, $(q_1, q_2)$ is the volumetric flow vector with reference to $(\partial x, \partial y)$ and $g$ represents the coefficient of the acceleration due to the gravity. The initial conditions at the starting time $t = 0$ are given while non-reflection boundary conditions will be considered in the implementation.

There are several numerical techniques for approximating (1), e.g., finite elements methods, finite volumes methods, spectral methods. In a classical way, the numerical solution of 2-D SW (shallow water) equations is affected by small oscillations called modes. In this paper, we present an accurate finite difference approach.

Thus far, two type of modes have been identified in the literature [3, 6] and mainly come from the approximation of the gravity waves. In one hand, when the waves travel with different speeds, dispersion occurs. In that case, the observed oscillations are called natural modes. Those unsteady wiggles do not present any risk for the quality of the numerical solution since they are inherent to the model itself. In the other hand, there are oscillations proper to the approximation of the convection terms. Such spurious modes are stationary and will accumulate over time to spoil the numerical solution. They are exclusively arising from the choice of the discretization grid (see [1]).

This paper investigates a finite difference scheme for an accurate numerical solution. The occurrence of the spurious modes is avoided by the use of the staggered grid $C$ of Arakawa. Our discretization approach involves a balance ratio of the grid nodes so that the natural modes present in the numerical solution can be eliminated. In others terms the included nodal parameter plays a limiter role.

This paper is organized as follow. In Sect. 2, we discretize the 2-D nonlinear SW model and we showcase the involvement of the nodal balance parameter. In Sect. 3, we analyze the stability with respect to the including parameters for a linearized version. Sect. 4 presents some numerical experiments to highlight the stability properties of the scheme.

2. Discretization Principle

Selecting a discretization principle for the purpose of solving the 2-D SW equations by finite difference requires special care to avoid spurious oscillations. In this
section, we focus on the discretization principle of the staggered grid $C$ of Arakawa: the components of the flow vector $(q_1, q_2)$ are computed at the interfaces of the cells according to the external normal, while the height $h$ variable is evaluated at the center of the cells. The 2-D domain $\Omega = [0, L_1] \times [0, L_2]$ is meshed uniformly as follows: $0 = x_0 < x_1 < \cdots < x_{n_1} = L_1$, $0 = y_0 < y_1 < \cdots < y_{n_2} = L_2$. A grid cell is denoted by $C_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ as depicted in the following figure.

![Figure 1. Staggered grid C of Arakawa](image)

### 2.1. Space discretization

We move from the continuous system (1) to the discrete space by seeking the solution $(h, q_1, q_2)$ at a finite number of nodes. For that, we consider a second order approximation according to the Arakawa grid $C$ principle (see Fig. 1). We denote by

$$h_{i+1/2,j+1/2} = h\left(i + 1/2\Delta x, j + 1/2\Delta y\right),$$

$q_{1i,j+1/2} = q_1\left(i\Delta x, (j + 1/2)\Delta y\right)$ and $q_{2i+1/2,j} = q_2\left((i + 1/2)\Delta x, j\Delta y\right)$.

We approximate the mass conservation equation for $i = 0, 1, \cdots, n_1 - 1$ and $j = 0, 1, \cdots, n_2 - 1$ as

$$\partial_t h_{i+1/2,j+1/2} + \frac{q_{i+1/2,j+1/2} - q_{i+1/2,j-1/2}}{\Delta x} + \frac{q_{i+1/2,j+1/2} - q_{i+1/2,j+1/2}}{\Delta y} = 0.$$

The approximation of the momentum equations for $i = 1, \cdots, n_1 - 1$ and $j = 1, \cdots, n_2 - 1$ is given by

$$\partial_t q_{1i,j+1/2} + g\left(\frac{h_{i-1/2,j+1/2} + h_{i+1/2,j-1/2}}{2}\right)\left(\frac{h_{i+1/2,j+1/2} - h_{i-1/2,j+1/2}}{\Delta x}\right) + \left(\frac{2q_{1i+1,j+1/2} - q_{1i-1,j+1/2}}{2\Delta x}\right)$$

$$+ \left(\frac{q_{2i+1/2,j+1/2} + q_{2i-1/2,j+1/2} + q_{2i+1/2,j+1/2} + q_{2i-1/2,j+1/2}}{4\left(h_{i+1/2,j+1/2} + h_{i-1/2,j+1/2}\right)}\right)\left(\frac{q_{1i+1/2,j+1/2} - q_{1i-1/2,j+1/2}}{2\Delta y}\right) = 0.$$
and
\[
\partial_t q_{i-1/2,j} + g \left( \frac{h_{i-1/2,j} + h_{i-1/2,j+1}}{2} \right) \left( \frac{h_{i-1/2,j} + h_{i-1/2,j+1} - h_{i-1/2,j} - h_{i-1/2,j+1}}{\Delta y} \right) \\
+ \frac{2 \left( q_{i-1/2,j+1/2} + q_{i+1/2,j+1/2} + q_{i+1/2,j+1} + q_{i-1/2,j+1} \right)}{\Delta x} \\
+ \left( \frac{q_{i-1/2,j} - q_{i-1/2,j+1}}{\Delta y} \left( \frac{2q_{i-1/2,j} - q_{i-1/2,j+1} - q_{i-1/2,j-1}}{2\Delta y} \right) \right) = 0.
\]

The discretization of the first momentum equation for \( j = 0, n_2 \) as well as the second momentum equation for \( i = 0, n_1 \) are easily deducible from the two above discrete equations.

### 2.2. Time discretization

We denote by \( h_{i+, j+\frac{1}{2}}^k, q_{i+, j+\frac{1}{2}}^k \) and \( q_{i-, j+\frac{1}{2}}^k \) the value of the discrete solution at time \( t^k \) respectively at nodes \( (i + \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta y \) (\( i\Delta x, (j + \frac{1}{2})\Delta y \)) and \((i - \frac{1}{2})\Delta x, j\Delta y \). The approximations of the time derivative terms in a interior cell \( C_{ij} \) are

\[
\frac{\partial}{\partial t} h_{i+, j+\frac{1}{2}}^{k+1} = \frac{1}{\Delta t} \left( \frac{q_{i+, j+\frac{1}{2}}^{k+1} - q_{i+, j+\frac{1}{2}}^k}{\Delta t} + \frac{1}{\Delta t} \left( \alpha h_{i+, j+\frac{1}{2}}^{k+1} - \alpha h_{i+, j+\frac{1}{2}}^k + (1 - 2\alpha)h_{i+, j+\frac{1}{2}}^k \right) \right),
\]

\[
\frac{\partial}{\partial t} q_{i+, j+\frac{1}{2}}^{k+1} = \frac{1}{\Delta t} \left( \frac{q_{i+, j+\frac{1}{2}}^{k+1} - q_{i+, j+\frac{1}{2}}^k}{\Delta t} + \frac{1}{\Delta t} \left( \alpha q_{i+, j+\frac{1}{2}}^{k+1} - \alpha q_{i+, j+\frac{1}{2}}^k + (1 - 2\alpha)q_{i+, j+\frac{1}{2}}^k \right) \right),
\]

\[
\frac{\partial}{\partial t} q_{i-, j+\frac{1}{2}}^{k+1} = \frac{1}{\Delta t} \left( \frac{q_{i-, j+\frac{1}{2}}^{k+1} - q_{i-, j+\frac{1}{2}}^k}{\Delta t} + \frac{1}{\Delta t} \left( \alpha q_{i-, j+\frac{1}{2}}^{k+1} - \alpha q_{i-, j+\frac{1}{2}}^k + (1 - 2\alpha)q_{i-, j+\frac{1}{2}}^k \right) \right).
\]

Introducing the implicit-explicit factor \( \theta \), it comes the following system

\[
\begin{cases}
q_{i+1}^{k+1} - Bq_i^k - (1 - \theta)\Delta tB_i^k q_i^{k+1} + \theta \Delta tB_i^{k+1} q_i^{k+1} = F_i^k \\
q_i^{k+1} - Bq_i^k - (1 - \theta)\Delta tB_i^k q_i^{k+1} + \theta \Delta tB_i^{k+1} q_i^{k+1} = F_i^k
\end{cases}
\]

\[
\begin{cases}
q_{i+1}^{k+1} - Cq_i^k - (1 - \theta)\Delta tC_i^k q_i^{k+1} + \theta \Delta tC_i^{k+1} q_i^{k+1} = F_i^k \\
q_i^{k+1} - Cq_i^k - (1 - \theta)\Delta tC_i^k q_i^{k+1} + \theta \Delta tC_i^{k+1} q_i^{k+1} = F_i^k
\end{cases}
\]

The matrices \( A, B \) and \( C \) depend only on the nodal balance parameter \( \alpha \) while \( A_i^k, B_i^k \) and \( C_i^k \) (with \( \kappa \) standing for \( k \) or \( k+1 \) and \( i = 1, 2, 3 \)) float according to the current solution \((h^k, q_i^k, q_j^k)\) and the unknown \((h^{k+1}, q_i^{k+1}, q_j^{k+1})\). The discrete system (2) can be recast in the following compact form

\[
N_{1, i}^{k+1} X_i^{k+1} + N_{2, i}^{k+1} X_i^k = F_i^k,
\]

where \( X_i^{k+1} = (h^{k+1}, q_i^{k+1}, q_j^{k+1}) \) is the unknown, the vector \( F_i^k \) contains the boundary conditions and \( N_{1, i}^{k+1} \) and \( N_{2, i}^{k+1} \) are two matrices depending on the state.
$X^{k+1}$ and $X^k$ respectively. According to the time discretization, it is suited to use the terminology implicit-dominance scheme for $\theta \in [0, 0.5]$ and explicit-dominance scheme when $\theta \in [0.5, 1]$. In this sequel $\theta = 0$ (resp. $\theta = 1$) corresponds to the full implicit (resp. explicit) scheme.

### 3. Stability Analysis

In this section, we analyze the stability of our numerical approach. For that, we follow the Von Neumann principle (see [5]) using the discrete Fourier transform. For sake of simplicity, we deal with the case $\Delta x = \Delta y$. If $f$ represents a hydrodynamical variable, we write

$$f = \frac{1}{2\pi} \int \left[ \frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x} \right] \exp(i_0 \Delta x m \cdot \zeta) \tilde{f}(\zeta) d\zeta,$$

where $\zeta \in \left[ \frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x} \right]^2$, $i_0$ is the unit complex ($i_0^2 = -1$) and $\tilde{f}$ stands for the Fourier transform of $f$,

$$\tilde{f}(\zeta) = \frac{(\Delta x)^2}{2\pi} \sum_{m \in \mathbb{Z}^2} \exp(-i m \cdot \zeta) f_m.$$

In a grid cell $C_{ij}$, we substitute the function $f$ in turns by $h^k$ for $m = (i + 1/2, j + 1/2)$, by $q^k_1$ for $m = (i, j + 1/2)$ and by $q^k_2$ for $m = (i + 1/2, j)$. We rewrite then the linearized discrete system in terms of Fourier transform $\hat{U}^k = \left( \hat{k}, \hat{q}^k_1, \hat{q}^k_2 \right)^T$ at time $t^k$ and $\hat{U}^{k+1} = \left( \hat{k}^{k+1}, \hat{q}^{k+1}_1, \hat{q}^{k+1}_2 \right)^T$ at time $t^{k+1}$ in the form

$$\hat{U}^{k+1} = C \hat{U}^k + i \left( \frac{\Delta t}{\Delta x} \theta \hat{M} \hat{U}^k + i_0 \left( \frac{\Delta t}{\Delta x} (1 - \theta) \hat{M} \hat{U}^{k+1} = 0. \right. \right.$$

For $\Phi_1 = \Delta x \zeta_1$ and $\Phi_2 = \Delta x \zeta_2$ in the interval $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ the matrices $C$ and $\hat{M}$ are given by

$$C = \begin{pmatrix}
\alpha \cos \Phi_1 + \alpha \cos \Phi_2 + 2(1 - 2\alpha) & 0 & 0 \\
0 & 2\alpha \cos \Phi_1 + 1 - 2\alpha & 0 \\
0 & 0 & 2\alpha \cos \Phi_2 + 1 - 2\alpha
\end{pmatrix}$$

and

$$\hat{M} = \begin{pmatrix}
0 & \sin \Phi_1 & \sin \Phi_2 \\
(g\tilde{v} - \tilde{u}^2) \sin \left( \frac{\Phi_1}{2} \right) - \tilde{u} \tilde{v} \sin \left( \frac{\Phi_2}{2} \right) & 2\tilde{u} \sin \Phi_1 + \tilde{v} \sin \Phi_2 & \tilde{v} \sin \Phi_2 \\
-\tilde{u} \tilde{v} \sin \frac{\Phi_1}{2} + (g\tilde{h} - \tilde{v}^2) \sin \frac{\Phi_2}{2} & \tilde{u} \sin \Phi_1 & 2\tilde{u} \sin \Phi_2 + \tilde{v} \sin \Phi_1
\end{pmatrix}.$$

We gather the similar quantities in (6) according to the factors $\hat{U}^{k+1}$ and $\hat{U}^k$ to get

$$G_1 \hat{U}^{k+1} = G_2 \hat{U}^k$$

where $G_1 = I_3 + i_0 (1 - \theta) \frac{\Delta t}{\Delta x} \hat{M}$ and $G_2 = C - i_0 \theta \frac{\Delta t}{\Delta x} \hat{M}$. 


Finally we recast (7) in the following amplifier form

\[ \hat{U}^{k+1} = G \hat{U}^k \quad \text{where} \quad G = G_1^{-1}G_2. \]  

(8)

The matrix \( G \) is called amplification matrix. The relation (8) reveals that going forward in time (one time step) is equivalent to multiplying the Fourier transform \( \hat{U}^k \) of the solution \( U^k \) by the amplification factor \( G \). Showing that the scheme is stable is equivalent to show that the sinusoidal input waves will not be amplified. Otherwise, the product of amplifications at every time-step will increase exponentially. In that case, the matrix \( G \) describes the degree of deformation of the output \( \hat{U}^{k+1} \) with respect to the input \( \hat{U}^k \). In this sense, the stability condition of our scheme is given by

\[ (C_s) : \quad \text{The spectral radius of } G \text{ is less than 1}. \]

---

**4. Numerical Experiments**

**4.1. Numerical illustration of the stability condition**

In this section, we discuss numerically the stability condition \((C_s)\) according to the implicit-explicit parameter \( \theta \) and the nodal balance parameter \( \alpha \). A classical way for studying the stability condition is to evaluate the Courant number for given parameters \( \alpha \) and \( \theta \). In our case, we fix the Courant number by the choice of the time-step \( dt \) and observe the stability variation according to those parameters. To compute the eigenvalues of the amplification matrix \( G \), we set \( \tilde{q}_1 = 1, \tilde{q}_2 = 1, \tilde{h} = 2 \) on every grid cell of size \( dx \times dy = 0.5 \times 0.5 \) with time step \( dt = 0.15 \).

**Lax-Friedrichs scheme.** For \( \alpha = 0.5 \), our scheme coincides with the Lax-Friedrichs one. Knowing the properties of the Lax-Friedrichs scheme, we observe the influence of the implicit-explicit factor \( \theta \) on the stability condition.

![Numerical results for different values of \( \theta \) (\( \alpha = 0.5 \))](image_url)
The above plots display the conditional stability of the scheme. For \( \theta \in [0, 0.5] \), we have an implicit-dominance stable scheme since all the eigenvalues live inside the unit circle. For higher value of \( \theta \) the scheme becomes unstable. However, the stability is re-acquirable by the proper choice of the parameter \( \alpha \). For instance \( \theta = 0.7 \) (figure (c)) becomes stable when the parameter \( \alpha \) is changed to 0.3.

**Crank-Nicholson scheme.** When \( \theta = 0.5 \), we have the Crank-Nicholson scheme. The stability condition is observed numerically through the following plots:

![Plots](image1.png)

For \( \theta = 0.5 \), one meet the classical form of the Crank-Nicholson scheme and its unconditional stability which is one of its main features. Inspection of the plots shows that this stability is conserving for values of \( \alpha \) less than 0.5.

**4.2. Numerical test**

In this subsection, we perform numerical experiment for the following stepwise linearized version of (3)

\[
N_1^k X^{k+1} + N_2^k X^k = \tilde{F}^k,
\]

using **Scilab** ([www.scilab.org](http://www.scilab.org)). We deal with homogeneous Dirichlet boundary conditions and the time step is computed according to the stability condition for the values \( \alpha = 0.5 \) and \( \theta = 0.8 \). The initial conditions are

\[
h^0(x, y) = \frac{20}{\sqrt{2\pi}} \exp\left(-0.15 (x^2 + y^2)\right) \quad \text{and} \quad (q_1^0, q_2^0) = (0, 0)
\]

in the square domain \([0, 15] \times [0, 15]\). We represent some states of the water elevation variation.
5. Conclusion

In this paper, we have presented a new finite difference scheme to simulate accurately the 2-D nonlinear shallow water equations. The nodal balance parameter including in the time discretization behaves as limiter since it provides a sort of diffusion to attenuate the convection dominance. This additional diffusion is explicit in the sense that large value of $\alpha$ amplifies the explicit feature of the scheme so that instability occurs when $\alpha > 0.5$.

6. Bibliographie